Chapter 29

Faraday’s Law

Electromagnetic Induction

- In the middle part of the nineteenth century Michael Faraday formulated his law of induction.
- It had been known for some time that a current could be produced in a wire by a changing magnetic field.
- Faraday showed that the induced electromotive force is directly related to the rate at which the magnetic field lines cut across the path.

Faraday’s Law

- Faraday's law of induction can be expressed as:
- The emf is equal to minus the change of the magnetic flux with time.

$$\xi = -\frac{d\Phi_B}{dt}$$

Magnetic Flux

- The magnetic flux is given by:

$$\Phi_B \equiv -\int \vec{B} \cdot d\vec{A}$$

Faraday's Law

- Therefore the induced emf can be expressed as:

$$\xi = -\frac{d}{dt}\int \vec{B} \cdot d\vec{A}$$

Faraday's Law for Simple Cases

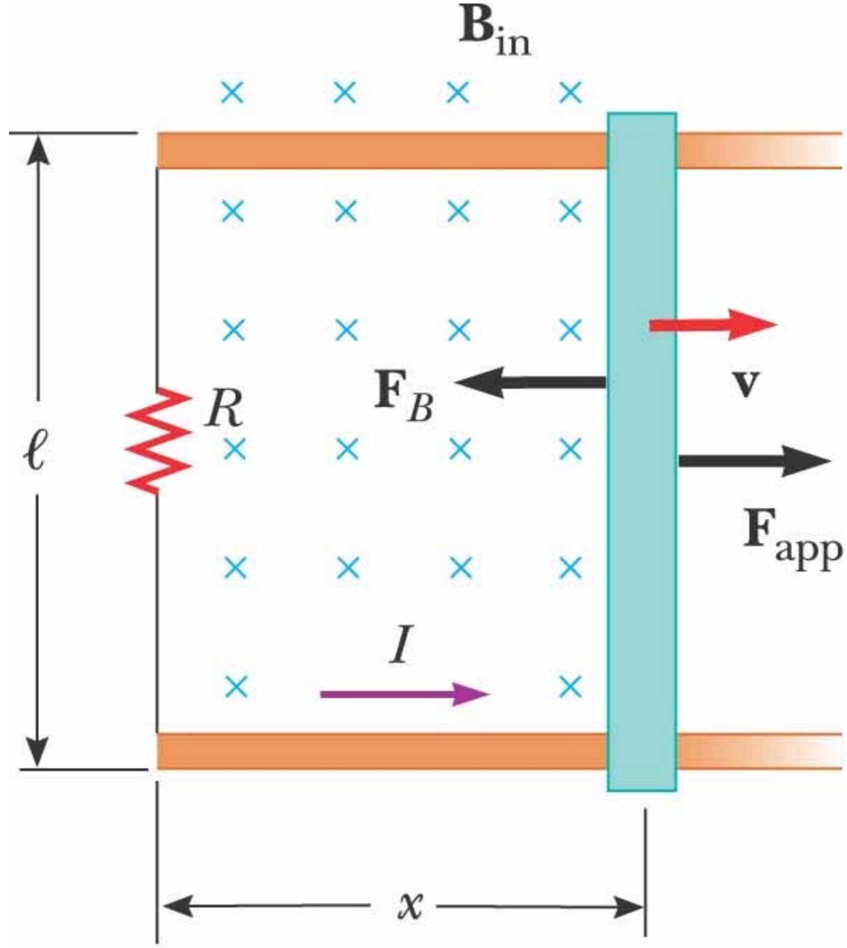
- If the magnetic field is spatially uniform and the area is simple enough then the induced emf can be expressed as:

$$\xi = B\frac{dA}{dt}$$

Example

- A rectangular loop of wire has an area equal to its width (x) times its length (L).
- Suppose that the length of the conducting wire loop can be arbitrarily shortened by sliding a conducting rod of length L along its width.
- Furthermore, suppose that a constant magnetic field is moving perpendicular to the rectangular loop.
- Derive an expression for the induced emf in the loop.

A sliding rod of length L in a magnetic field.



(a)

Solution

- If we move the conductor along the conducting wires the area of the loop that encloses the magnetic field is changing with time.
- The amount of change is proportional to the velocity that which we move the rod.

Solution cont.

- The induced emf depends on the magnitude of the magnetic field and the change of the area of the loop with time.

$$\xi = B\frac{dA}{dt} = BL\frac{dx}{dt}$$

Solution cont.

- Since the length L remains constant the width changes as:

$$\frac{dx}{dt} = \text{v}$$

- Then the induced emf is:

$$BL\text{v}$$

Example

- Suppose the rod in the previous figure is moving at a speed of 5.0 m/s in a direction perpendicular to a 0.80 T magnetic field.
- The conducting rod has a length of 1.6 meters.
- A light bulb replaces the resistor in the figure.
- The resistance of the bulb is 96 ohms.

Example cont.

- Find the emf produced by the circuit,
- the induced current in the circuit,
- the electrical power delivered to the bulb,
- and the energy used by the bulb in 60.0 s.

Solution

- The induced emf is given by Faraday’s law:

$$\xi = \mathrm{v}BL = \left(5.0\,m/s\right)\left(0.8\,T\right)\left(1.6\,m\right) = 6.4\,V$$

Solution cont.

- We can obtain the induced current in the circuit by using Ohm’s law.

$$I = \frac{\xi}{R} = \frac{6.4V}{96\Omega} = 0.067\,A$$

Solution cont.

- The power can now be determined.

$$P = I\xi = (0.067\,A)(6.4V) = 0.43W$$

Solution cont.

- Since the power is not changing with time, then the energy is the product of the power and the time.

$$E = Pt = (0.43W)(60.0\,s) = 26\,J$$

The Emf Induced by a rotating Coil

- Faraday's law of induction states that an emf is induced when the magnetic flux changes over time.
- This can be accomplished
- by changing the magnitude of the magnetic field,
- by changing the cross-sectional area that the flux passes through, or
- by changing the angle between the magnetic field and the area with which it passes.

The Emf Induced by a rotating Coil

- If a coil of *N* turns is made to rotate in a magnetic field then the angle between the B-field and the area of the loop will be changing.
- Faraday's law then becomes:

$$emf = -\frac{d\Phi_B}{dt} = -NAB\left(\frac{d\cos\theta}{dt}\right) = NAB\sin\theta\frac{d\theta}{dt}$$

Angular Speed

- The angular speed can be defined as:

$$\omega = \frac{d\theta}{dt}$$

- Then integrating we get that:

$$\theta = \omega t$$

The Emf Induced by a rotating Coil

- Substitution of the angular speed into our relation for the emf for a rotating coil gives the following:

$$\xi = NAB\omega \sin \omega t$$

Example

- The armature of a 60-Hz ac generator rotates in a 0.15-T magnetic field.
- If the area of the coil is 2×10^{-2} m^2 , how many loops must the coil contain if the peak output is 170 V?

Solution

- The maximum emf occurs when the sin ωt equals one. Therefore:

$$\xi_{\max} = NBA\omega$$

- Furthermore, we can calculate the angular speed by noting that the angular frequency is:

$$\omega = 2\pi f = 2\pi(60\,Hz) = 377\,s^{-1}$$

Solution cont.

- The number of turns is then:

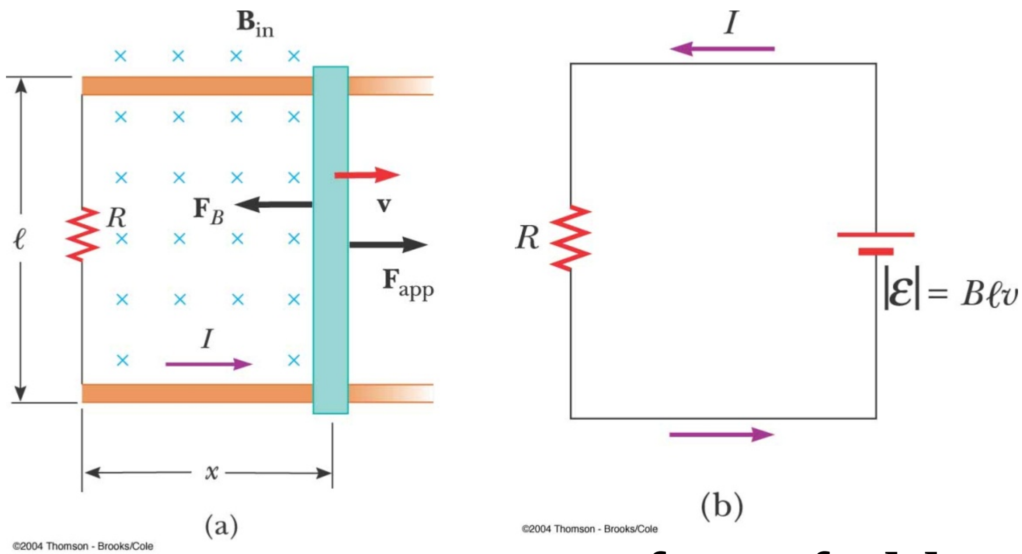
$$N = \frac{\xi_{\max}}{BA\omega} = \frac{170V}{(0.15T)(2.0\times10^{-2}m^2)(377s^{-1})} = 150$$

Another Form

- Suppose the have a stationary loop in a changing magnetic field.
- Then, since the path of integration around the loop is stationary, we can rewrite Faraday's law.

$$\xi = \oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

Sliding Conducting Bar



- A bar moving through a uniform field and the equivalent circuit diagram
- Assume the bar has zero resistance
- The work done by the applied force appears as internal energy in the resistor *R*

Lenz’s Law

- Faraday’s law indicates that the induced emf and the change in flux have opposite algebraic signs
- This has a physical interpretation that has come to be known as **Lenz’s law**
- Developed by German physicist Heinrich Lenz

Lenz’s Law, cont.

- **Lenz’s law**: *the induced current in a loop is in the direction that creates a magnetic field that opposes the change in magnetic flux through the area enclosed by the loop*
- The induced current tends to keep the original magnetic flux through the circuit from changing

Induced emf and Electric Fields

- An electric field is created in the conductor as a result of the changing magnetic flux
- Even in the absence of a conducting loop, a changing magnetic field will generate an electric field in empty space
- This induced electric field is nonconservative
  - Unlike the electric field produced by stationary charges

Induced emf and Electric Fields

- The induced electric field is a nonconservative field that is generated by a changing magnetic field
- The field cannot be an electrostatic field because if the field were electrostatic, and hence conservative, the line integral of $\mathbf{E} \cdot d\mathbf{s}$ would be zero and it isn't

Another Look at Ampere's Law

- Ampere’s Law states the following:

$$\oint \vec{B} \cdot d\vec{s} = \mu_o I$$

Another Look at Ampere's Law

- Thus we can determine the magnetic field around a current carrying wire by integrating around a closed loop that surrounds the wire and the result should be proportional to the current enclosed by the loop.

Another Look at Ampere’s Law

- What if however, we place a capacitor in the circuit?
- If we use Ampere's law we see that it fails when we place our loop in between the plates of the capacitor.
- The current in between the plates is zero sense the flow of electrons is zero.

- What do we do now?

Maxwell’s Solution

- In 1873 James Clerk Maxwell altered ampere's law so that it could account for the problem of the capacitor.

Maxwell's Solution cont.

- The solution to the problem can be seen by recognizing that even though there is no current passing through the capacitor there is an electric flux passing through it.
- As the charge is building up on the capacitor, or if it is oscillating in the case of an ac circuit, the flux is changing with time.

Maxwell’s Solution cont.

- Therefore, the expression for the magnetic flux around a capacitor is:

$$\oint \vec{B} \cdot d\vec{s} = \mu_o \varepsilon_o \frac{d\Phi_E}{dt}$$

Maxwell's Solution cont.

- If there is a dielectric between the plates of a capacitor that has a small conductivity then there will be a small current moving through the capacitor thus:

$$\oint \vec{B} \cdot d\vec{s} = \mu_o I + \mu_o \varepsilon_o \frac{d\Phi_E}{dt}$$

Maxwell’s Solution cont.

- Maxwell proposed that this equation is valid for any arbitrary system of electric fields, currents, and magnetic fields.
- It is now known as the Ampere-Maxwell Law.
- The last term in the previous equation is known as the displacement current.

Magnetic Flux Though a Closed Surface

- Mathematically, we can express the features of the magnetic field in terms of a modified Gauss Law:

$$\oint_S \vec{B} \cdot d\vec{A} = 0$$

Magnetic Flux Though a Closed Surface cont.

- The magnetic field lines entering a closed surface are equal to the number of field lines leaving the closed surface.
- Thus unlike the case of electric fields, there are no sources or sinks for the magnetic field lines.
- Therefore, there can be no magnetic monopoles.

What Have We Learned So Far?

- Gauss's Law for Electricity.

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_o}$$

- Gauss's Law for Magnetism.

$$\oint_S \vec{B} \cdot d\vec{A} = 0$$

What Have We Learned So Far?

- Faraday's Law.

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

- Maxwell-Ampere's Law.

$$\oint \vec{B} \cdot d\vec{s} = \mu_o I + \mu_o \varepsilon_o \frac{d\Phi_E}{dt}$$

- The four previous equations are known as Maxwell's equations.

Differential Calculus

- We can express Maxwell's equations in a different form if we introduce some differential operators.

$$\nabla a = \frac{\partial a_x}{\partial x}\hat{i} + \frac{\partial a_y}{\partial y}\hat{j} + \frac{\partial a_z}{\partial z}\hat{k}$$

$$\nabla \bullet \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Gauss's Theorem, or Green's Theorem, or Divergence Theorem

- Consider a closed surface S forming a volume V.
- Suppose the volume is of an incompressible fluid such as water.
- Imagine that within this volume are an infinite amount of infinitesimal faucets spraying out water.

- The water emitted from each faucet could be represented by the vector function **F**.
- The total water passing through the entire surface would be:

$$\oint_S \vec{F} \cdot d\vec{A}$$

- This quantity of water must equal the total amount of water emitted by all the faucets.
- Since the water is diverging outward from the faucets we can write the total volume of water emitted by the faucets as:

$$\int_V \left(\nabla \cdot \vec{F}\right) d\tau$$

- Therefore, the total water diverging from the faucets equals the total water passing through the surface.

$$\int_V \left(\nabla \cdot \vec{F} \right) d\tau = \oint_S \vec{F} \cdot d\vec{A}$$

- This is the divergence theorem, also known as Gauss's or Green's theorem.

Stoke's Theorem

- Consider the curl of the velocity tangent to the circle for a rotating object.

$$\nabla \times \vec{\mathrm{v}} = \nabla \times \left(\vec{\omega} \times \vec{r} \right)$$

- We can rewrite this with the following vector identity:

$$\vec{A}\times\vec{B}\times\vec{C}=\left(\vec{A}\cdot\vec{C}\right)\vec{B}-\left(\vec{A}\cdot\vec{B}\right)\vec{C}$$

- The curl then becomes:

$$\nabla\times\left(\vec{\omega}\times\vec{r}\right)=\left(\nabla\cdot\vec{r}\right)\vec{\omega}-\left(\vec{\omega}\cdot\nabla\right)\vec{r}$$

- In Cartesian coordinates we see that the first term is:

$$\vec{\omega}\left(\nabla \cdot \vec{r}\right) = \vec{\omega}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3\vec{\omega}$$

- The second term is:

$$\left(\vec{\omega}\cdot\nabla\right)\vec{r}=\left(\omega_x\frac{\partial}{\partial x}+\omega_y\frac{\partial}{\partial y}+\omega_z\frac{\partial}{\partial z}\right)\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=\omega_x\hat{i}+\omega_y\hat{j}+\omega_z\hat{k}$$

$$=\vec{\omega}$$

- Therefore, we get the following:

$$\nabla \times \vec{v} = 2\vec{\omega}$$

- The curl of the velocity is a measure of the amount of rotation around a point.

- The integral of the curl over some surface, *S* represents the total amount of swirl, like the vortex when you drain your tub.
- We can determine the amount of swirl just by going around the edge and finding how much flow is following the boundary or perimeter, *P*.
- We can express this by the following:

$$\int_S \left(\nabla \times \vec{F}\right) \cdot d\vec{A} = \oint_P \vec{F} \cdot d\vec{s}$$

Differential Form of Maxwell's Equations

- We can now write Maxwell's equations as:

$$\nabla \cdot \vec{E} = \frac{\rho_f}{\varepsilon_o} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell’s Equations

- The previous equations are known as Maxwell's equations for a vacuum.
- The equations are slightly different if dielectric and magnetic materials are present.

Maxwell’s Equations

- The differential form of Maxwell's when dielectric and magnetic materials are present are as follows:

$$\nabla \cdot \vec{D} = \rho_f$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{H} = J_f + \frac{\partial \vec{D}}{\partial t}$$

One More Differential Operator

- Consider the dot product of the gradient with itself.

$$\nabla \cdot \nabla = \nabla^2$$

- This is a scalar operator called the Laplacian.
- In Cartesian coordinates it is:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$